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Connection Between the Asymptotic Behavior and the Sign of the Discontinuity in One-Dimensional Dispersion Relations*

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Starting from one-dimensional dispersion relations (either fixed-transfer or partial-wave), information on the sign of absorptive part which follows from unitarity, and in some cases analyticity in the Mandelstam ellipse in the t plane and polynomial boundedness, we derive various consequences of physical interest, e.g., a high-energy lower bound on the forward scattering amplitude, the minimum fluctuation of the sign of the discontinuity across the left-hand cut in the partial-wave dispersion relations, etc. In the derivations, the positiveness of the absorptive part plays an essential role by allowing us to construct a Herglotz function which has a well-known asymptotic behavior.

I. INTRODUCTION

WE want to show in this paper that if a scattering amplitude satisfies a dispersion relation and if one has some information about the sign of the discontinuity of this amplitude across its cuts, then the scattering amplitude under consideration cannot decrease arbitrarily fast at infinity. Two main applications will be presented:

(a) Forward dispersion relations, and with some supplementary assumptions fixed transfer dispersion relations. Here the sign of the discontinuity is given by the unitarity condition. It turns out that in general the forward scattering amplitude cannot decrease faster than $1/s^2(\ln s)^{1/2}$. However, under certain conditions involving only information on *low energies*, this can be improved and the scattering amplitude may be shown to decrease slower than $1/(\ln s)^{1/2}$. By combination with analyticity assumptions with respect to the momentum transfer some weak lower bounds can be deduced for the absorptive part of the scattering amplitude itself. At the same time some results on the behavior of the scattering amplitude below threshold are derived by using crossing symmetry.

(b) Partial-wave dispersion relations. Here only the sign of the discontinuity on the physical cut is known. Then, instead of getting information on the high-energy behavior, we will impose this high-energy behavior from unitarity together with the threshold behavior and get, in this way, information about the left-hand cut discontinuity. It will turn out that the minimum number of oscillations of the left-hand cut discontinuity increases as the angular momentum increases.

The technique we shall use involves properties of Hergoltz functions and reduction of non-Herglotz functions to Herglotz functions. So we shall start with a few mathematical considerations.

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II. HERGLOTZ AND RELATED FUNCTIONS

We remind that a Herglotz function[1] $H(z)$ is a function analytic in $\mathrm{Im}z>0$ such that $\mathrm{Im}H(z)>0$ for $\mathrm{Im}z>0$. It admits the integral representation

$$H(z)=A+Bz+\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Im}H(x)(1+zx)}{(1+x^2)(x-z)}dx, \qquad (1)$$

with $B\geq 0$, $\mathrm{Im}H(x)\geq 0$, and the condition that

$$\int_{-\infty}^{+\infty}\frac{\mathrm{Im}H(x)dx}{1+x^2}\text{ converges}. \qquad (2)$$

In addition, we have, since $-1/H(z)$ is a Herglotz function,

$$\int_{-\infty}^{+\infty}\frac{\mathrm{Im}H(x)}{|H(x)|^2(1+x^2)}dx\text{ convergent}. \qquad (3)$$

From representation (1) it is clear that in complex directions $\epsilon<\arg z<\pi-\epsilon$

$$C/|z|<|H(z)|<C|z|. \qquad (4)$$

However, for physical applications, we need to have a statement on the behavior of $|H(x)|$ for x real, and Eqs. (2), (3), and (4) are insufficient in this respect.

It is clear that along the real axis we should only expect behavior of the type (4) for some averaged function. This is most conveniently done by introducing a new function

$$G(z)=\frac{1}{z}\int_0^z H(z')dz'. \qquad (5)$$

It is easy to see, by using a straight line as integration path for (5) that $G(z)$ is again a Herglotz function.

[1] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943); cf., especially p. 23.

Hence we have, from (3)

$$\int_A^\infty \frac{dx}{x} \frac{\int_0^x \operatorname{Im}H(x')dx'}{\left|\int_0^x H(x')dx'\right|^2} \text{ convergent.} \qquad (6)$$

Now the origin of coordinates may certainly be chosen such that for $x>0$ $\operatorname{Im}H(x)$ is not identically zero. The quantity $\int_0^x \operatorname{Im}H(x')dx'$ is a nondecreasing function of x. Therefore, the convergence of (6) implies that

$$\int_A^\infty \frac{dx}{x} \frac{1}{\left[\int_0^x |H(x')|dx'\right]^2} \text{ converges.}$$

This provides us with the desired condition for $x>x_0>0$

$$\int_A^x |H(x')|dx' > C(\ln x)^{1/2}, \qquad (7)$$

which implies

$$\limsup_{x\to+\infty} x|H(x)|(\ln x)^{1/2} = +\infty\,. \qquad (8)$$

The same statement holds, of course, for $x\to-\infty$, with corresponding reversal of sign.

Equations (7) and (8) can be interpreted as follows: Either $\operatorname{Im}H(x)$ decreases less fast than $1/x$ at infinity and then we do not learn anything new, or $\operatorname{Im}H(x)$ decreases very fast, but then $\operatorname{Re}H(x)$ cannot decrease arbitrarily fast. In particular, if we take $\operatorname{Im}H(x)$ to be zero outside a finite interval, we easily see, from representation (1), that $\operatorname{Re}H(x)$ behaves like $1/x$ at large distances.

Next, we want to extend these considerations to functions which are not Herglotz functions but which are analytic in a twice-cut plane, with two cuts from $-\infty$ to A and from B to $+\infty$, bounded in the entire plane (including the cuts) by a polynomial in $|z|$, real in the sense $F(z)=F^*(z^*)$, and such that on the cuts $\operatorname{Im}F(x+i\epsilon)>0$. This has been done already by Symanzik[2] for the case where only one cut is present.

Let us start by showing that such a function has only a finite number of zeros in the twice-cut plane. Let $F<|z|^N$ for $|z|>z_0$. F has pairs of complex zeros at say z_i and z_i^* and real zeros x_j for $A\le x_j\le B$. If the number of these zeros is infinite, we can divide F by

$$\prod_{i=1}^{p}(z-z_i)(z-z_i^*)\prod_{j=1}^{2q}(z-x_j)\,,$$

such that $2p+2q>N+2$. The new function G obtained in this way decreases at least like $1/|z|^2$ at large distances, has a positive discontinuity on both cuts, and can be represented by an unsubtracted dispersion relation.

$$G(z)=\frac{1}{\pi}\int_{-\infty}^{A}\frac{\operatorname{Im}G(x)dx}{x-z}+\frac{1}{\pi}\int_{B}^{\infty}\frac{\operatorname{Im}G(x)dx}{x-z}\,.$$

Clearly, G cannot decrease faster than $1/z$ in complex directions. Therefore, the total number of zeros is finite and certainly less or equal to $N+2$.

Then two cases occur:

(i) The number of real zeros is even. Then we remove all the zeros:

$$F(z)=\Pi(z-z_i)(z-z_j^*)\Pi(z-x_j)H(z)\,.$$

Then $H(z)$ has a positive discontinuity across both cuts and no zeros. H can now easily be shown to be a Herglotz function, using, for instance, the method of Symanzik.[2]

(ii) The number of real zeros is odd. Then we remove all complex zeros and all real zeros except one. If the new function $G(z)$ obtained in this way is such that it has a positive derivative at the remaining zero, then again, it can be shown that it is a Herglotz function. If it has a negative derivative at $z=x_R$ (remaining x_R) we divide it by $(z-x_R)^2$, introducing in this way a pole at $z=x_R$ with negative residue, i.e., a *positive* δ-function contribution to the discontinuity of the new function, which again will be shown to be a Herglotz function.

We note that in *all* cases, for large $|z|$ we have $|F(z)|\simeq|z|^p|H(z)|$, where $H(z)$ is a Herglotz function, and $p\ge0$. Therefore, conditions (7) and (8) apply as well to $F(z)$ as to $H(z)$:

$$\int_B^x |F(x')|dx' > C(\ln x)^{1/2},$$

$$\int_{-x}^{A} |F(x')|dx' > C(\ln x)^{1/2}, \qquad (9)$$

and

$$\limsup_{x\to\pm\infty}|x|\;|F(x)|(\ln|x|)^{1/2}=+\infty\,. \qquad (10)$$

If, instead of having $\operatorname{Im}F(x)>0$ on the cuts, we allow for the possibility of *isolated* zeros, $\operatorname{Im}F(x)\ge0$, we can work with the averaged function

$$\bar{F}(z)=\frac{1}{\Delta}\int_z^{z+\Delta}F(z')dz'\,,$$

which will hane a positive definite discontinuity on the cuts from $-\infty$ to $A+\Delta$ and from $B-\Delta$ to $+\infty$. Then Eqs. (9) and (10) hold for $\bar{F}(z)$. It follows immediately that Eqs. (9) and (10) hold for $F(z)$ itself.

The next step is to consider a function $F(z)$ with two cuts, from $-\infty$ to A and from B to $+\infty$, with the same reality property, but whose discontinuity changes sign ν times from $x=-\infty$ to $x=+\infty$. Then we can define a new function

$$G(z)=\prod_{k=1}^{\nu}(z-z_k)F(z)\,,$$

[2] K. Symanzik, J. Math. Phys. 1, 249 (1960), Appendix B.

which, on the two cuts has a positive or zero discontinuity. This function satisfies conditions (9) and (10) and hence we get

$$\int_A^x |x'|^\nu |F(x')| dx' > (\ln x)^{1/2}, \tag{11}$$

$$\limsup_{x\to\pm\infty} |x|^{\nu+1} (\ln|x|)^{1/2} |F(x)| = +\infty\,. \tag{12}$$

Our mathematical apparatus is now ready and we can turn to physical applications.

III. FIXED TRANSFER (IN PARTICULAR, FORWARD) DISPERSION RELATIONS

We consider a scattering process

$$A+B\to A+B \tag{I}$$

coupled by crossing to

$$A+\bar{B}\to A+\bar{B} \tag{II}$$

for which one can write a forward dispersion relation. It is further assumed that below the thresholds for both processes the absorptive part vanishes. More precisely,

$$F(s,0)=P_N(s)+\frac{s^N}{\pi}\int_{(M_A+M_B)^2}^{\infty}\frac{\mathrm{Im}F_{\mathrm{I}}(s',0)ds'}{s'^N(s'-s)} +\frac{u^N}{\pi}\int_{(M_A+M_B)^2}^{\infty}\frac{\mathrm{Im}F_{\mathrm{II}}(u',0)du'}{u'^N(u'-u)}\,, \tag{13}$$

where s is the square of the c.m. energy, and

$$u=2M_A{}^2+2M_B{}^2-s\,.$$

From unitarity, we have

$$\mathrm{Im}F_{\mathrm{I}}(s',0)>0 \quad \mathrm{Im}F_{\mathrm{II}}(u',0)>0$$

and then, in the variable s, F is analytic in a twice cut plane with a *positive* right-hand cut discontinuity and a *negative* left-hand cut discontinuity. We are, therefore, in the case $\nu=1$ of last section, and we get

$$\int_{s_0}^{s} s'|F(s',0)|ds' > (\ln s)^{1/2}$$

$$\limsup_{s\to\infty} s^2(\ln s)^{1/2}|F(s,0)|=+\infty\,. \tag{14}$$

The scattering amplitude, therefore, cannot decrease arbitrarily fast. In the special case where a power behavior is assumed

$$|F(s,0)|\sim s^\alpha,$$

we get

$$\alpha\geq -2\,. \tag{15}$$

We are perfectly aware of the fact that this result is very weak and that nobody would in any case believe that the scattering amplitude could decrease so fast. However, from a purely theoretical point of view this is better than nothing at all.

If fixed transfer dispersion relations are assumed to hold also in the unphysical region

$$0\leq t\leq 4\mu^2\,,$$

with

$$t=-2k^2(1-\cos\theta)\,,$$

$$k \text{ c.m. momentum}$$

$$\cos\theta \text{ c.m. scattering angle}\,,$$

then in this region we still have

$$\mathrm{Im}F_{\mathrm{I}}(s',t)>0\,,\quad \mathrm{Im}F_{\mathrm{II}}(u',t)>0\,,$$

so that the result still holds.

Now in the negative t region we do not know *a priori* the sign of $\mathrm{Im}F(s,t)$. However, we can obviously establish a connection between the number of oscillations of $\mathrm{Im}F(s,t)$ for fixed t and the asymptotic behavior of $\mathrm{Im}F(s,t)$. Take, for instance, for simplicity, a self-conjugate process ($B\equiv\bar{B}$). Then if $F(s,t)\sim s^{\alpha(t)}f(t)$ for $s\to+\infty$, we get

$$\alpha(t)\geq -2-2\nu(t)\,, \tag{16}$$

where ν is the number of oscillations of $\mathrm{Im}F(s,t)$ from $s=2\mu^2-t/2$ to $+\infty$. ν is by assumption finite, since $\mathrm{Im}F(s,t)$ has a definite sign in the asymptotic region. So the high-energy behavior is in part conditioned by information on the low-energy behavior in the *same* channel. Equation (16) is, of course, weaker than the Gribov condition[3] $\alpha(t)>-1$ but it is established under much weaker assumptions.

We want to present more detailed arguments to reinforce our statement that the low-energy behavior of a scattering amplitude gives information on the high-energy behavior. Let us return to the forward scattering amplitude $A+B\to A+B$ with $\bar{B}=B$. Then it is advantageous to use the symmetric variable

$$z=(s-M_A{}^2-M_B{}^2)^2$$

which maps the upper half-plane of the s variable on the whole z plane with a single cut from $z=z_0=(2M_AM_B)^2$ to $+\infty$. Assume now that the conditions for the validity of the Froissart bound are satisfied,[4] namely the analyticity of $F(s,t)$ in t up to $t=4\mu^2$ and the boundedness of $F(s,t)$ by polynomials in s for fixed t in the region of analyticity.

Then we can write the scattering amplitude in a unique way, since $F(s,0)<s\ln^2 s$

$$F(s,0)\equiv G(z,0)=A+\frac{z}{\pi}\int_{z_0}^{\infty}\frac{\mathrm{Im}G(z',0)dz'}{z'(z'-z)}\,, \tag{17}$$

This formula shows clearly that $G(z,0)$ itself is a Herglotz function. From this follows that for real $z<z_0$ all the derivatives of G with respect to z are positive. Hence,

[3] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 239 (1962). See also, P. G. O. Freund and R. Oehme, Phys. Rev. **129**, 2361 (1963).

[4] A. Martin, Lecture Notes of the Scottish Universities Summer School, 1963 (to be published).

if $G(z_0,0)$ is negative $[F((M_A+M_B)^2,0)<0]$, $G(z,0)$ is negative for $z\leq z_0$. Hence, the function $G(z,0)/(z-z_1)$, with $z_1<z_0$ is again a Herglotz function. Hence, we get

$$\int_A^x \frac{|G(x',0)|}{x'}dx' > (\ln x)^{1/2},$$

and

$$\limsup (\ln x)^{1/2}|G(x,0)| = +\infty,$$

or

$$\limsup |F(s,0)|(\log s)^{1/2} = +\infty. \tag{18}$$

So the knowledge that $G(z_0,0)$ is negative (in other terms the knowledge that the zero-energy S-wave scattering length is negative), is sufficient to allow us to gain two powers of s in the high-energy lower bound of $|F(s,0)|$.

If the zero-energy scattering length is positive, it is still possible to get some further information. Let us again change variable and use

$$y=z-z_0=(s-M_A{}^2-M_B{}^2)^2-4M_A{}^2M_B{}^2;$$

then

$$F(s,0)=G(z,0)=H(y,0)=A'+\frac{y}{\pi}\int\frac{\mathrm{Im}H(y',0)dy'}{y'(y'-y)}. \tag{19}$$

The subtraction can be made at $y=0$ because at threshold $\mathrm{Im}H(y,0)\sim\sqrt{y}$. Then for $y<0$ we get, given C arbitrary, positive

$$H(y,0)<H(0,0)+\frac{y}{\pi}\int_0^C\frac{\mathrm{Im}H(y',0)dy'}{y'(y'-y)}$$

and, in particular,

$$H(-\infty,0)<H(0,0)-\frac{1}{\pi}\int_0^C\frac{\mathrm{Im}H(y',0)dy'}{y'}. \tag{20}$$

So, if we can find a number C such that the right-hand side of (20) is negative, this tells us that in $-\infty<y<0$ $H(y,0)$ has one zero, with positive slope, then $H(y,0)/(y-y_0)$ is again a Herglotz function, and we are back to result (18), i.e., an average decrease of $|F(s,0)|$ not faster than $(\ln s)^{-1/2}$. This necessitates only the knowledge of the zero-energy scattering length and of the total cross section in the *low-energy* region. For instance, for equal masses μ, scattering length a_0 we get the general condition for the validity of (18):

$$\mu a_0-\frac{1}{8\pi^2}\int_{4\mu^2}^C\frac{\sigma_t(s)(s-2\mu^2)ds}{(s-4\mu^2)^{1/2}s^{1/2}}<0. \tag{21}$$

We expect that if the process under consideration has a resonance, the inequality (21) will be satisfied by taking C above the resonance energy. So, if (21) holds, the forward scattering amplitude lies between the asymptotic limits $(\ln s)^{-1/2}$ and $s\ln^2 s$. Whether the gap between these two extreme cases is large or not is a matter of taste and of scale. If we just count the powers of s we see that the interval is not so big.

Let us first mention now two theoretical implications of (14) and/or (18). It has been shown[5,6] that if the scattering amplitude is larger than s^{-N} in the forward direction:

(i) The elastic cross section σ_e satisfies

$$\sigma_e>(\sigma_t)^2(\ln s)^{-2}.$$

(ii) The large angle scattering amplitude cannot be uniformly less than $\exp(-Cs^{1/2}\ln s)$ if Mandelstam representation holds. It is clear that now the assumption about the forward scattering amplitude is no longer necessary.

We want now to show that when one supplements (14) and (18) by analyticity assumptions in t $[F(s,t)$ analytic in t up to $t=4\mu^2$ and bounded by a polynomial in $s]$ one can deduce lower bounds on $\mathrm{Im}F(s,0)$ from the lower bounds on $|F(s,0)|$. Indeed, these analyticity assumptions imply that only $Cs^{1/2}\ln s$ partial waves contribute effectively to the scattering amplitude. The error committed by neglecting higher waves may be shown to be less than s^{-N}, where N can be made very large by taking C big enough. Hence using Schwarz inequality together with the unitarity condition for partial waves, $1\geq\mathrm{Im}f_l\geq|f_l|^2$, we have:

$$\left|F(s,0)-O\left(\frac{1}{s^N}\right)\right|^2<\left|\frac{s^{1/2}}{k}\sum_0^{Cs^{1/2}\ln s}(2l+1)f_l(s)\right|^2$$

$$<C^2s\ln^2 s\left(\frac{s^{1/2}}{k}\right)\left(\sum_0^\infty(2l+1)|f_l(s)|^2\right)$$

$$<C^2\frac{s^{1/2}}{k}s\ln^2 s\,\mathrm{Im}F(s,0).$$

Hence if $|F(s,0)|$ is larger than $(\ln s)^{-1/2}s^{-2}$, we get

$$\mathrm{Im}F(s,0)>(\ln s)^{-3}s^{-5}. \tag{22}$$

If $|F(s,0)|$ is larger than $(\ln s)^{-1/2}$, we get

$$\mathrm{Im}F(s,0)>(\ln s)^{-3}s^{-1}. \tag{23}$$

IV. THE SCATTERING AMPLITUDE BELOW THRESHOLD

We have seen that the zero transfer scattering amplitude expressed as a function of $z=(s-2\mu^2)^2$ (in the equal masses case $M_A=M_B=\mu$), is a Herglotz function. However, in a previous paper[7] we have shown that for any fixed $t>0$ inside the analyticity domain in t, the same conclusion can be drawn. Here, specifically, we shall treat the case of $\pi^0\pi^0$ scattering. Notice that the coupling of the scattering process with the reaction $\pi^0\pi^0\to\pi^+\pi^-$ does not affect our conclusions, because it does not affect the positiveness of $\mathrm{Im}F(s,0)$ for $s>4\mu^2$.

So, if we define $z=(s-2\mu^2+t/2)^2$ we have, according

[5] A. Martin, Nuovo Cimento 29, 993 (1963).
[6] F. Cerulus and A. Martin, Phys. Letters 8, 80 (1964).
[7] Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).

to Ref. 7:

$$F(s,t)=G(z,t)=A(t)+\frac{z}{\pi}\int_{z_0(t)}^{\infty}\frac{\operatorname{Im}G(z',t)dz'}{z'(z'-z)} \quad (24)$$

in $0<t<4\mu^2$, with $z_0(t)=(2\mu^2+t/2)^2$, $\operatorname{Im}G(z',t)>0$. Hence for $z<z_0(t)$ we have

$$(d/dz)^n G(z,t)>0$$

and changing variables; for $0\leq t\leq 4\mu^2$

$$2\mu^2-t/2<s<4\mu^2$$

$$\left(\frac{d}{ds}\right)^n F(s,t)>0\,,\quad \textit{for fixed } t. \quad (25)$$

Since we have complete symmetry of the amplitude in all three channels, we can interchange variables and in this way get information on the scattering amplitude in the triangle $s<4\mu^2$, $t<4\mu^2$, $u<4\mu^2$. First we show that the symmetry point

$$s=t=u=4\mu^2/3$$

is an absolute minimum of $F(s,t,u)$ at least inside the triangle. Indeed inside the triangle, two out of the three variables s, t, u are positive. Let s and t be positive. If $s>t>u$, let us distinguish two cases:

(i) $t>4\mu^2/3$, which implies $s>4\mu^2/3$

$$\begin{aligned}F(s,t,u)=F(s,t,u)-F[s,\,4\mu^2/3,\,(8\mu^2/3)-s]\\ +F[s,\,4\mu^2/3,(8\mu/3)-s]-F(4\mu^2/3,4\mu^2/3,4\mu^2/3)\\ +F(4\mu^2/3,4\mu^2/3,4\mu^2/3).\end{aligned}$$

Hence, $F(s,t,u)>F(4\mu^2/3,4\mu^2/3,4\mu^2/3)$.

(ii) $t<4\mu^2/3$, which implies $u<4\mu^2/3$

$$\begin{aligned}F(s,t,u)=F(s,t,u)-F[(8\mu^2/3)-t,\,t,\,(4\mu^2/3)]\\ +F[(8\mu^2/3)-t,\,t,\,4\mu^2/3]-F(4\mu^2/3,4\mu^2/3,4\mu^2/3)\\ +F(4\mu^2/3,4\mu^2/3,4\mu^2/3).\end{aligned}$$

In both cases we get

$$F(s,t,u)>F(4\mu^2/3,4\mu^2/3,4\mu^2/3). \quad (26)$$

This is true in all the triangle $s<4\mu^2$, $t<4\mu^2$, $u<4\mu^2$ by permutations over s, t, and u. We believe that this is a result of some use, since in many approximate schemes the threshold amplitude, i.e., the s-wave scattering length, is taken to be equal to the value of the scattering amplitude at the symmetry point. Inequality (26) indicates in which direction the error goes.

It is also easy to see that inside the triangle F increases along any straight line originating at the symmetry point, by the same kind of argument.

It is slightly more delicate to show that if we define the S-wave amplitude as

$$\phi_0(s)=\frac{2}{s-4\mu^2}\int_{4\mu^2-s}^{0}F(s,\,t,\,4\mu^2-s-t)dt; \quad (27)$$

then for

$$2\mu^2<s<4\mu^2$$
$$d\phi_0(s)/ds>0. \quad (28)$$

Owing to the symmetry, $\phi_0(s)$ may be rewritten as

$$\phi_0(s)=2\int_0^{1/2}F[s,\,x(4\mu^2-s),\,(4\mu^2-s)(1-x)]dx.$$

Then

$$\frac{d\phi_0(s)}{ds}=2\int_0^{1/2}\left(\left(\frac{\partial F}{\partial s}\right)_t-x\left(\frac{\partial F}{\partial t}\right)_s\right)dx.$$

We have

$$\left(\frac{\partial F}{\partial s}\right)_t>0 \quad \text{for} \quad t>0 \text{ and } s>2\mu^2>2\mu^2-t/2\,,$$

$$\left(\frac{\partial F}{\partial t}\right)_s<0 \quad \text{for} \quad s>0 \text{ and } t<2\mu^2-s/2\,,$$

which proves (28). We think that (28) might have some usefulness in removing part of the ambiguity in the solutions of N/D equations for the s wave.

From (27), it also follows almost obviously

$$\begin{aligned}\phi_0(s)<F(s,\,0,\,4\mu^2-s)\,,\quad 0\leq s<4\mu^2\,,\\ \phi_0(4\mu^2)=F(4\mu^2,0,0)\,.\end{aligned} \quad (29)$$

Inequality (29) shows for instance that a strong D-wave resonance, which will enhance the imaginary part of the forward scattering amplitude and hence induce through relation (17) or (19) a strong energy dependence of $F(s,0)$ in $0<s<4\mu^2$, will at the same time induce a still stronger energy dependence of the S wave in $0<s<4\mu^2$ and hence in all the low-energy region.

V. THE LEFT-HAND CUT IN PARTIAL-WAVE DISPERSION RELATIONS

We want now to apply the considerations of Sec. II to a partial-wave amplitude, analytic in a twice-cut plane, with cuts

$$-\infty\rightarrow -A \quad 4\mu^2\rightarrow +\infty.$$

(For simplicity, we take the equal mass case.) The partial-wave scattering amplitude, normalized as

$$\phi_l(s)=[s/(s-4\mu^2)]^{1/2}e^{i\delta l}\sin\delta_l(s)\,, \quad (30)$$

where δ_l is real in the elastic region, is assumed to be bounded by a polynomial. We know from general unitarity condition that the right-hand cut discontinuity is positive or zero. The sign of the left-hand cut discontinuity is unknown. Let ν_ϕ be the number of oscillations of the left-hand cut. What we want to do is to get some information on ν_ϕ.

We shall now set the requirement that $\phi_l(s)$ has the correct threshold behavior. Hence,

$$\phi_l(s)=(s-4\mu^2)^l\psi_l(s)\,, \quad (31)$$

where $\psi_l(s)$ has a finite limit for $s\rightarrow 4\mu^2$. $\psi_l(s)$ is again a polynomial bounded function with a discontinuity which changes sign ν_ψ times. It is clear that

$$\nu_\phi = \nu_\psi \quad \text{for } even\ l$$

and

$$|\nu_\phi - \nu_\psi| = 1 \quad \text{for } odd\ l.$$

More precisely, in the *odd l* case

$$\text{if} \quad \mathrm{Im}\phi(-A-y+i\epsilon) > 0$$
$$\nu_\psi = \nu_\phi + 1;$$
$$\text{if} \quad \mathrm{Im}\phi(-A-y+i\epsilon) < 0$$
$$\nu_\psi = \nu_\phi - 1.$$

Now we apply inequality (12) to $\psi_l(s)$ and we get

$$\limsup_{s\to\infty} s^{\nu_\psi+1}(\ln s)^{1/2} s^{-l} |\phi_l(s)| = +\infty. \tag{32}$$

However, we know from unitarity that $|\phi_l(s)| < 1+\epsilon$ for $s > s_0$. Hence, condition (32) can only be satisfied if we have

$$\nu_\psi + 1 - l \geq 0. \tag{33}$$

Condition (33) can be slightly improved by a more careful use of unitarity

$$\mathrm{Im}\phi_l(s) \geq [(s-4\mu^2)/s]^{1/2} |\phi_l(s)|^2,$$

on the right-hand cut. Hence, asymptotically,

$$\mathrm{Im}\psi_l(s) \geq s^{-l} |\psi_l(s)|^2.$$

Now following the procedure of Sec. II, we multiply ψ_l by a polynomial of degree ν_ψ to obtain a function $\chi_l(s)$ such that on the right cut, for large s

$$\mathrm{Imp}\chi_l(s) \geq s^{\nu_\psi - l} |\chi_l(s)|^2.$$

Next, we replace $\chi_l(s)$ by an averaged function

$$\bar{\chi}_l(s) = \frac{1}{\Delta} \int_s^{s+\Delta} \chi_l(s')ds'.$$

Then it is not difficult to show, using Schwarz inequality, that

$$\mathrm{Im}\bar{\chi}_l(s) > s^{\nu_\psi - l} |\bar{\chi}_l(s)|^2.$$

Then the zeros of $\bar{\chi}_l(s)$ are factored out in order to get a Herglotz function $H_l(s)$. Clearly, according to the study made in Sec. II, $H_l(s)$ will still satisfy for large s

$$\mathrm{Im}H_l(s) > s^{\nu_\psi - l} |H_l(s)|^2.$$

At this point we use Eq. (6) directly and Schwartz inequality

$$\int_A^\infty \frac{ds}{s} \frac{\int_A^s \mathrm{Im}H_l(s')ds'}{\left|\int_A^s H_l(s')ds'\right|^2} > C \int_{4\mu^2}^\infty \frac{ds}{s} \frac{1}{\int_A^s s'^{\,l-\nu_\psi} ds'}.$$

The right-hand side of this equation should converge. For $\nu_\psi = l-1$, which is the lower limit predicted by (33); it gives

$$\int_{4\mu^2}^\infty \frac{ds}{s \ln s}$$

which is obviously divergent. Hence, we must replace (33) by

$$\nu_\psi \geq l. \tag{34}$$

It is easy, according to the rules indicated above, to make the corresponding statement for the discontinuity of the original amplitude $\phi_l(s)$. In particular, if the near left-hand cut discontinuity is given by the projection of the lth wave of a one-particle pole in the transfer channel, we get

$$\nu_\phi = \nu_\psi \quad \text{for even } l\text{; hence,} \quad \nu_\phi \geq l$$
$$\nu_\phi = \nu_\psi + 1 \text{ for odd } l\text{; hence,} \quad \nu_\phi \geq l+1,$$

so that the P-wave left-hand cut has at least two changes of sign (counted by comparison to the right-hand cut).

We wish to make two comments about this result: First, it is clear that the structure of the left-hand cut becomes increasingly complicated as the angular momentum becomes large. Second, the consistency requirements we have obtained, which are *independent* of the number of subtractions, should be kept in mind when one deals with consistency problems, in particular, "bootstraps." Should the left-hand cut not satisfy our conditions, then either the threshold behavior or unitarity at high energy are mistreated.

VI. CONCLUDING REMARKS

We think that the few examples we have given show how powerful the tool of Herglotz functions is in dispersion theory and how important are the positivity requirements due to unitarity. This was already felt in a previous paper[7] where the number of subtractions of fixed transfer dispersions was limited in this way.

One may, of course, feel disappointed that these general requirements on scattering amplitude lead to rather weak conditions, most of which everybody was prepared to believe without proof. However, it seems to us to be a necessary task to explore bit-by-bit the *rigorous* consequences of analyticity, unitarity, and crossing. Who knows if some day one will not be able to reassemble the pieces of the puzzle?

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